Chaotic dynamics and nonlinear time-series analysis

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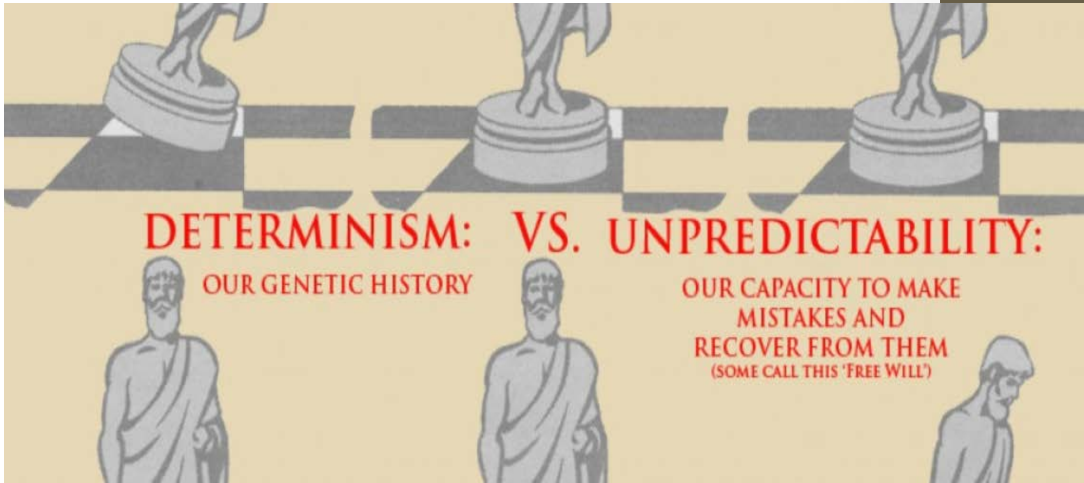
Chaos theory is the study of apparently ***random*** or unpredictable behaviour in systems governed by ***deterministic*** laws. ---- *Encyclopedia Britannica.*

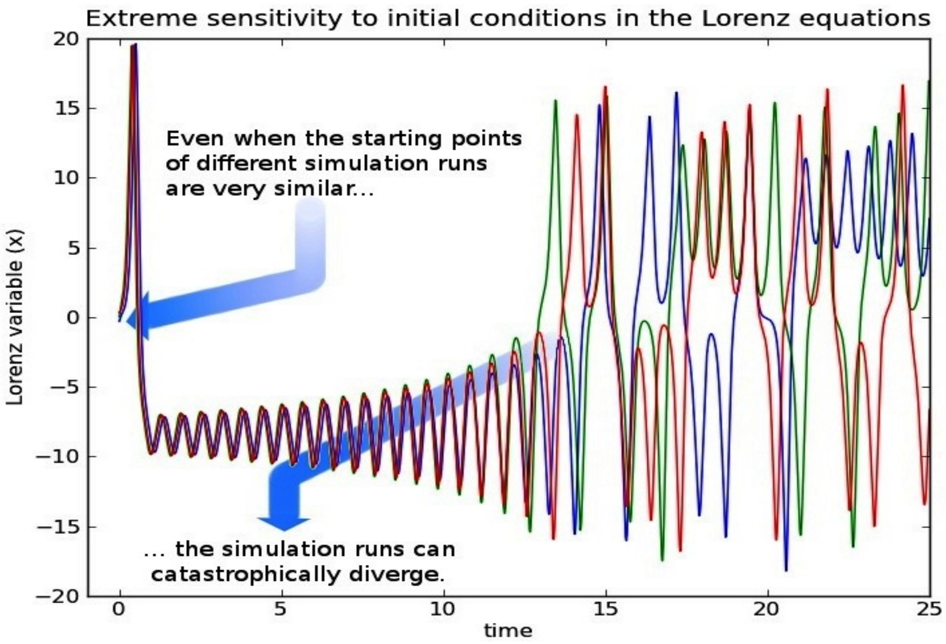
Deterministic chaos suggests a paradox because it connects two different notions: -

a) Determinism

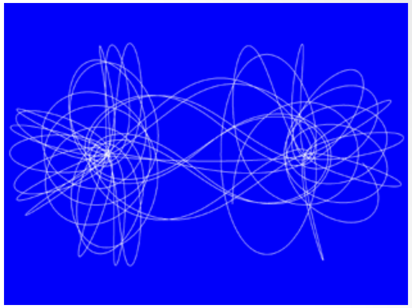
b) Randomness



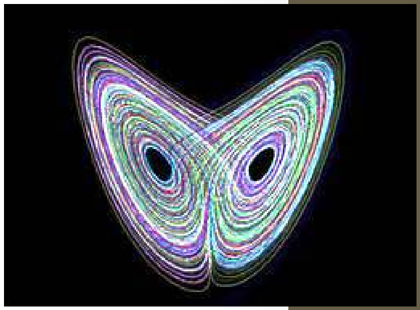
In deterministic dynamical systems, ***deterministic chaos*** is an unstable aperiodic behavior, which shows ***sensitive dependence on initial conditions.***



Examples of chaotic dynamical systems are:



- The solar system (Three-body problem)



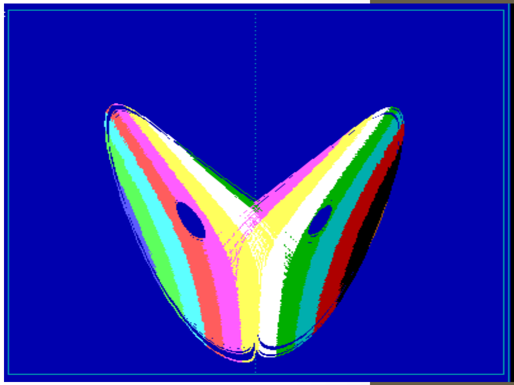
- Atmosphere (Butterfly effect)
- Population dynamics (Logistic map)
- Human body (Cardiac arrhythmias)
- Double pendulum
- Power electronics circuits

The **Butterfly Effect** is a term coined by Edward Lorenz. It highlights the possibility that small causes may have vital effects.

$$\dot{x} = \sigma(y - x)$$

$$\dot{y} = rx - y - xz$$

$$\dot{z} = xy - bz$$



Lorenz attractor

In chaos theory, the butterfly effect is the **sensitive dependence on initial conditions** in which a small change in one state of a **nonlinear deterministic** system can result in large differences in a later state.

1976: Robert May's famous review on Logistic Map

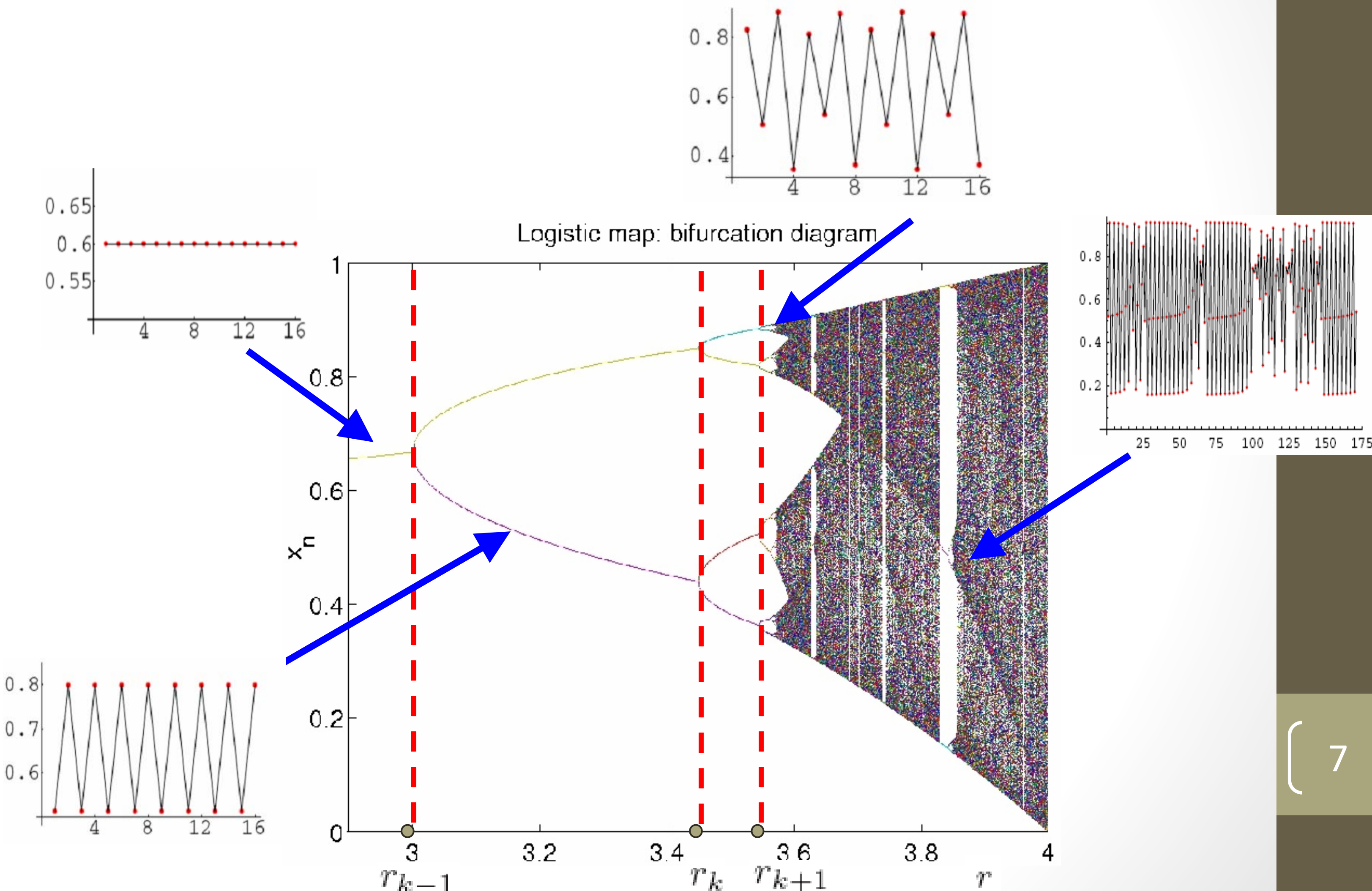
- The logistic map is a polynomial mapping.

- It is often cited as a classic example of how complex, chaotic behaviour can arise from very simple nonlinear dynamical equations.

$$x_{n+1} = r x_n (1 - x_n)$$

$$x \in [0,1] \qquad 0 < r \leq 4$$

Time-Series plots & Bifurcation Diagram of Logistic Map



Mile Stones

1890 – Henri Poincare – found non-periodic orbits while studying three body problem

1927 - Van der Pol – observed chaos in radio circuit.

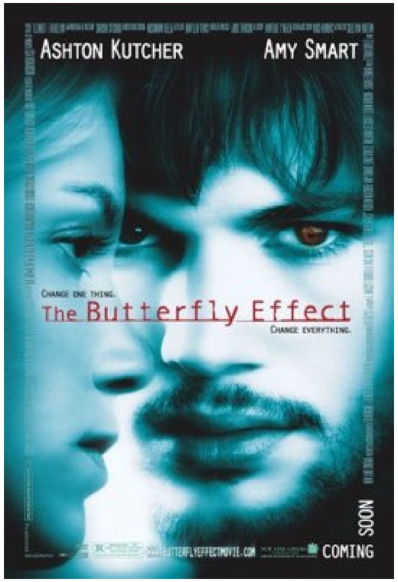
1960 – Edward Lorenz – "Butterfly effect"

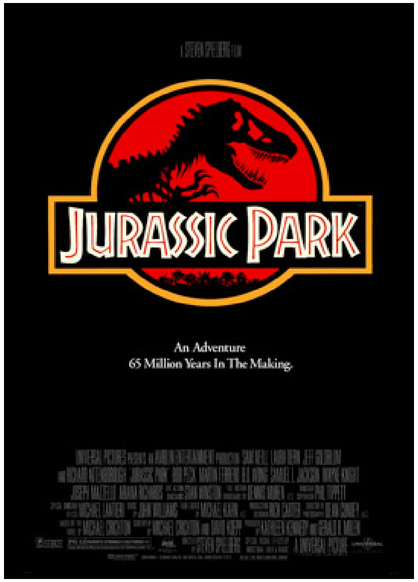
1975 – Li and Yorke coined the term "chaos"

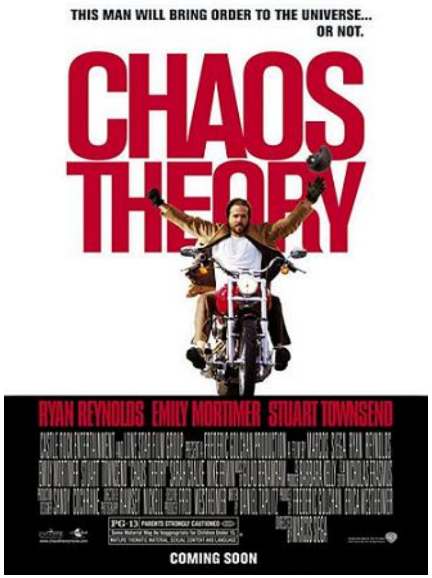
1970s – Mandelbrot – "The Fractal Geometry of Nature"

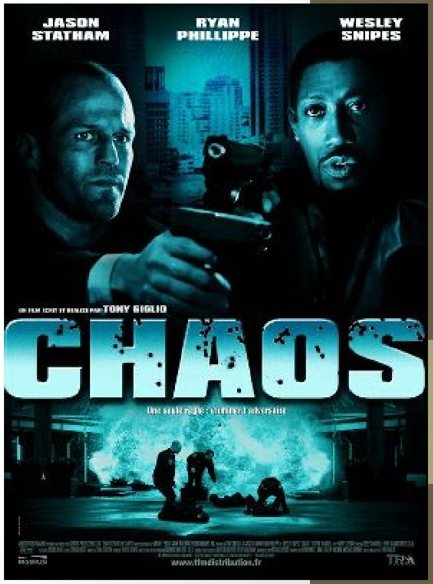
1976 – Robert May – wrote his famous review on "Logistic Map"

Chaos theory has been mentioned in works of literature and movies.





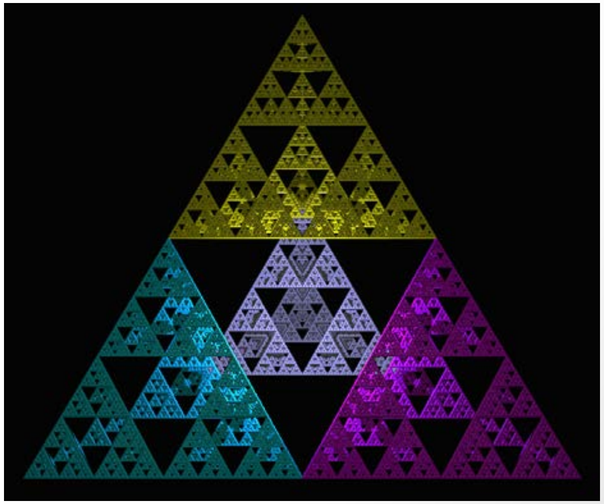




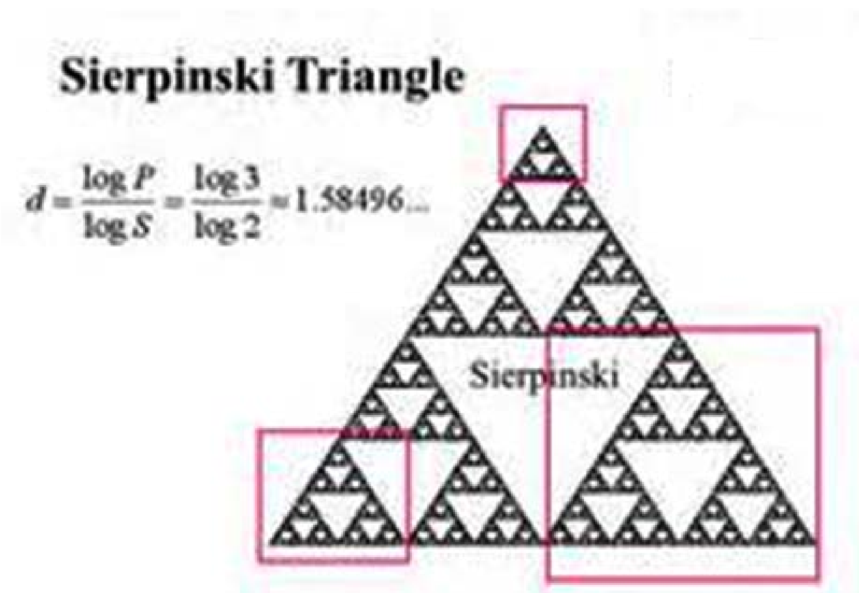
Fractals are infinitely self-similar, iterated, and detailed structures having fractal dimension.

Romanesco broccoli is strikingly fractal in nature.

Self-similarity is symmetry across all scales.



Fractals appear self-similar under different degrees of magnification, so they have their own dimension; ***fractal dimension.***



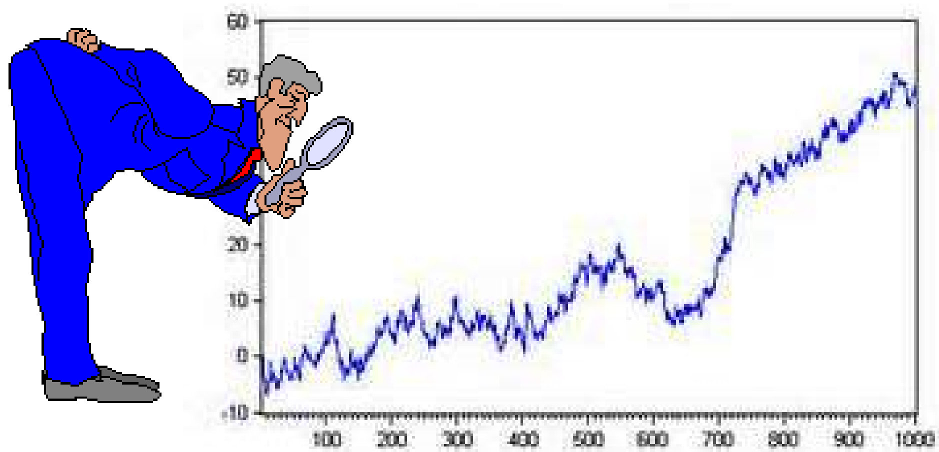
Chaotic dynamics are quantified:

- To distinguish chaotic behavior from random (noisy) behavior.

- To determine the variables required to model the dynamics of the system.

- To understand the changes in the dynamical behavior of the system.

The most direct link between chaos theory and the real world is the analysis of time-series.



It has been widely used in different problems, which either lack a mathematical model or lack access to multiple variables.

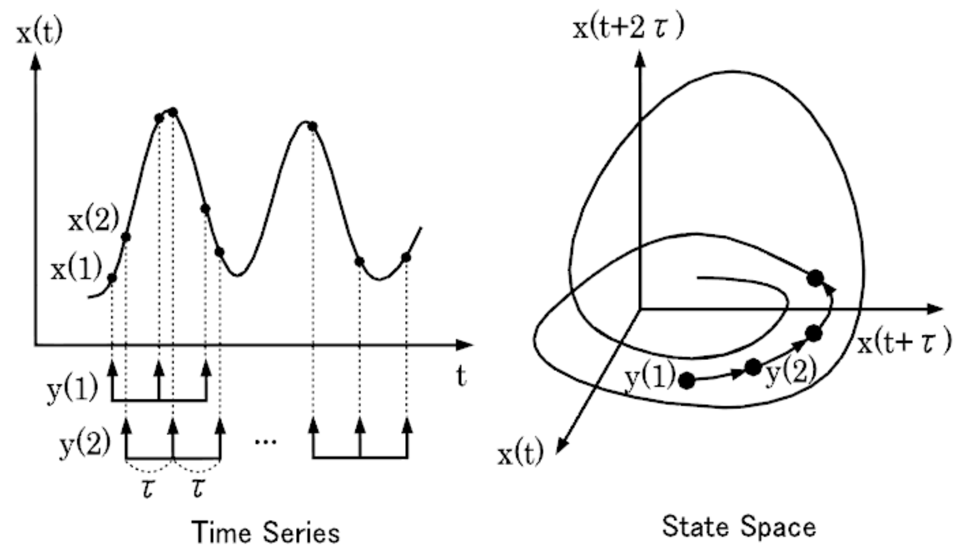
Nonlinear time-series analysis usually consists of these steps:

(i) Reconstruction of the system dynamics in the state-space using **time-delay embedding**.

(ii) Quantification of the reconstructed attractor using nonlinear measures.

- Correlation dimension
- Lyapunov exponent

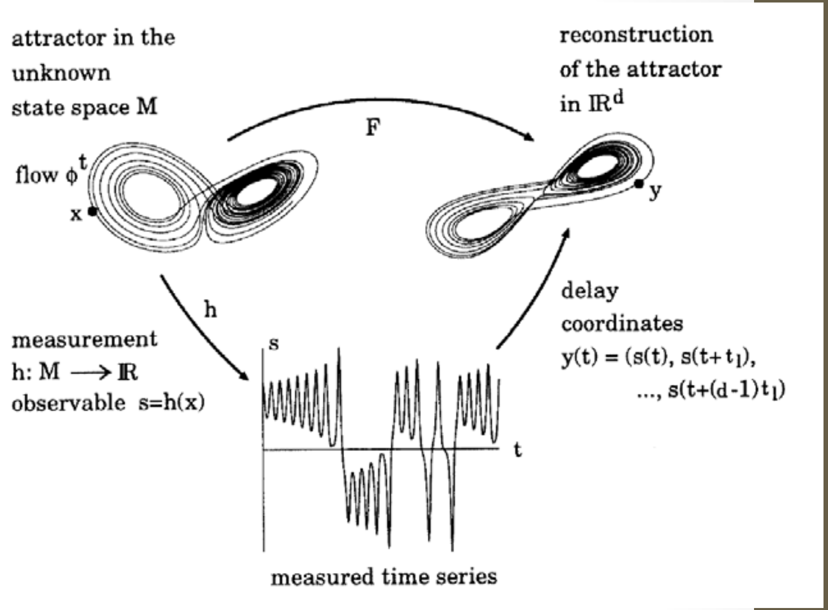
Embedding is a mapping from one-dimensional space to an n-dimensional space.



Takens proved that there is one-to-one correspondence between the reconstructed state-space and the original dynamical space.

Packard and Takens separately introduced state-space reconstruction.

They demonstrated that a vector space, which is comparable to the actual phase-space can be reconstructed from scalar data.



It is possible to conserve geometrical invariants like the correlation dimension and the Lyapunov exponent of the attractor.

Let a time-series

$$x_1, x_2, x_3, \ldots, x_N$$

be embedded into an m-dimensional state-space by the delay vectors.

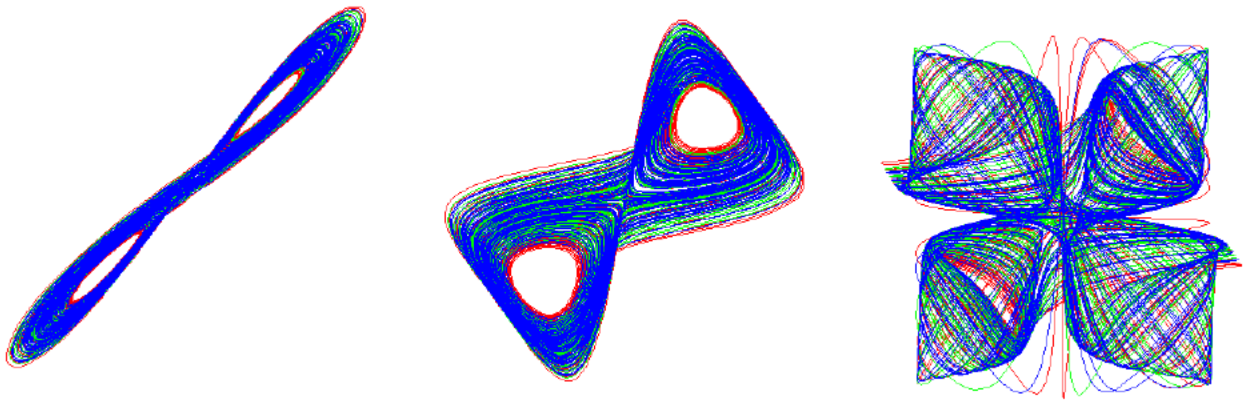
In the state-space, a point is given as:

$$y(\mathrm{t}) = x_t, x_{t-\tau}, x_{t-2\tau}, \ldots, x_{t-(\mathrm{m}-1)\tau}$$

Where ‘m’ is the embedding dimension and τ is the time delay or time lag.

According to Takens' embedding theorem, embedding can be achieved at almost any time lag , provided m is sufficiently large.

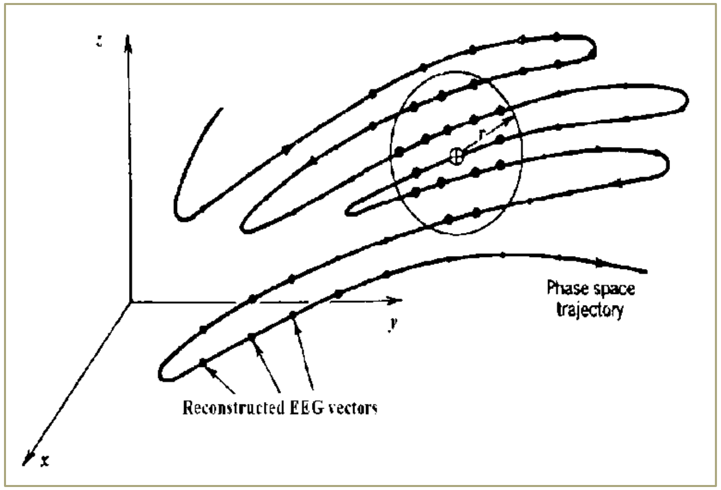
But extreme values of τ (either too large or too small) will produce problems in reconstructing attractor.



a) b) c)

Reconstructed phase-spaces for Lorenz system with embedding dimension, m = 3 and different values of time delays = 3, 17, and 100

Correlation dimension characterizes the fractal structure of the attractor.



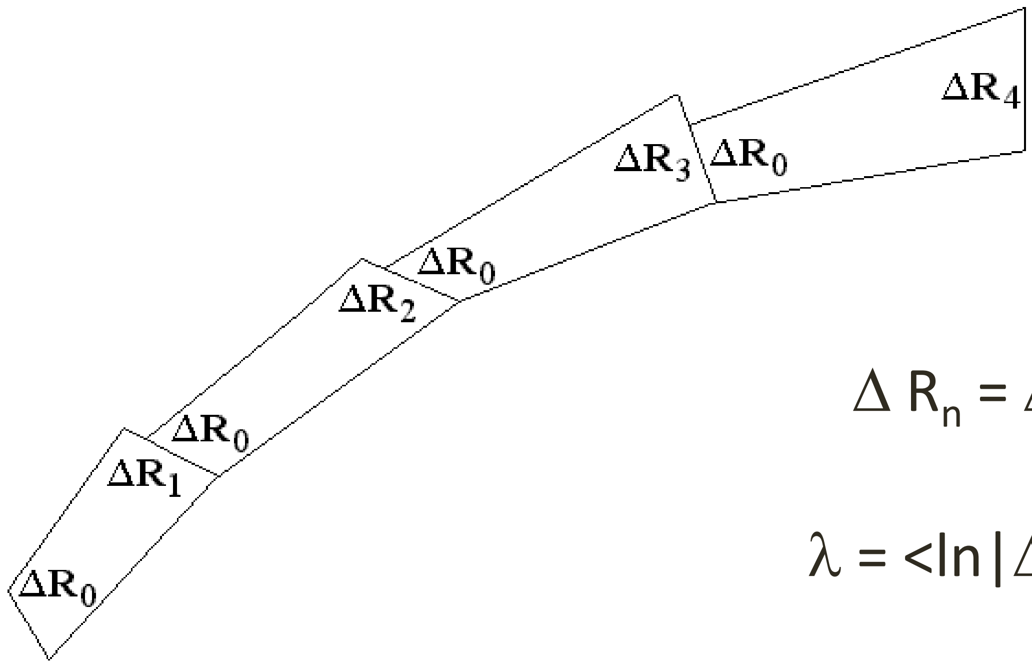
$N(r) \propto r^{v}$

$v = d\log N(r)/d\log(r)$

It is an estimate of the dimensionality of the space occupied by a set of random points.

- If v integer -> the attractor is a regular geometric object
- If v noninteger -> the attractor is a fractal

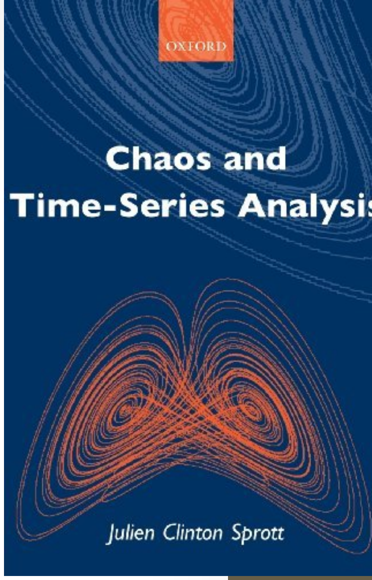
Lyapunov exponent quantifies rate of divergence of neighboring trajectories in the reconstructed phase-space.



$$\Delta R_n = \Delta R_0 e^{\lambda n}$$

$$\lambda = \langle \ln|\Delta R_n/\Delta R_0| \rangle$$

- For stable fixed points,; $\lambda < 0$,
- For noise; $\lambda = \infty$,
- For chaotic attractors; $\lambda > 0$



"When the present determines the future, but the approximate present does not approximately determine the future.“

-Edward Lorenz